

Mathematics 100A Quiz 3

Due: Friday 6 November 2020

Instructions: This is a closed note, closed book quiz. Please write clearly and fully explain your solutions. You have 30 minutes from when you begin to read the questions to complete it.

1. This question has two parts.

(a) Denote by S the set of $n \times n$ matrices with integer entries and nonzero determinant, i.e.,

$$S = \text{GL}_n(\mathbf{R}) \cap M_n(\mathbf{Z}) = \{g \in M_n(\mathbf{Z}) : \det(g) \neq 0\}.$$

Is S a group? If so, prove that S is a group; if not, explain why S is not a group.

(b) Suppose $A \in \text{GL}_n(\mathbf{R})$ is a fixed matrix. Denote by T the set of $g \in \text{GL}_n(\mathbf{R})$ with ${}^t g A g = A$, i.e.,

$$T = \{g \in \text{GL}_n(\mathbf{R}) : {}^t g A g = A\}.$$

Is T a group? If so, prove that T is a group; if not, explain why T is not a group.

Proof. Here are the two parts:

(a) The set S is not a group as it is not closed under inverses. For example, S contains the diagonal matrix $\text{diag}(2, 1, 1, \dots, 1)$ but not its inverse $\text{diag}(\frac{1}{2}, 1, 1, \dots, 1)$.

(b) The set T is a group. Indeed, it clearly contains the identity. It is closed under multiplication because if g_1, g_2 are in T then ${}^t(g_1 g_2) A (g_1 g_2) = {}^t g_2 ({}^t g_1 A g_1) g_2 = {}^t g_2 A g_2 = A$. It is also closed under inverses because if $g \in T$, then ${}^t g A g = A$, so moving the g 's to the other side of the equation one obtains $A = {}^t(g^{-1}) A g^{-1}$, so $g^{-1} \in T$.

□

2. Suppose G and G' are groups and $\varphi : G \rightarrow G'$ is a surjective group homomorphism. Suppose moreover that G has order 81, i.e., $|G| = 81$. What are the possible values for the order of G' ?

Proof. The possible orders for G' are 1, 3, 9, 27, 81. Indeed, because φ is surjective,

$$|G'| = |\varphi(G)| = [G : \ker(\varphi)] = \frac{|G|}{|\ker(\varphi)|}$$

is a factor of $|G|$. As $81 = 3^4$, the factors of 81 are 1, 3, $3^2 = 9$, $3^3 = 27$ or $3^4 = 81$. By taking G to be cyclic of order 81, one can show that all the possible values in $\{1, 3, 9, 27, 81\}$ can actually occur for the order of G' . However, to receive full credit, one does not need to prove this last fact. □